Lecture 36 (The one that explains how a ray in G/K might give a parabolic in G)

G simp conn semisimple $K \subset G$ max cpt.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ $\quad \mathfrak{a} \subset \mathfrak{p}$ Cartan subsp.

$Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{a}$ for $\mathfrak{m} \subset \mathfrak{k}$.

$\Phi(\mathfrak{g}, \mathfrak{k}, \mathfrak{a})$ = root sys. in $\mathfrak{a}^*$ $\quad \boxed{\Phi^+ \subset \Phi}$ Choice $\quad \Delta \subset \Phi^+$ determined.

$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus \mathfrak{g}_\alpha$. $\quad \mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$

Minimal parabolic $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Other parabolic: $S \subset \Delta$. $\Phi_S^- := \mathbb{Z}$-span $S \cap \Phi^-$

Then $\mathfrak{q}_S = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \bigoplus_{\alpha \in \Phi_S^-} \mathfrak{g}_\alpha$. $\rightsquigarrow$ parab subgroup. (connected, self-norm.)

What these give for $SL_n\mathbb{R}$, $SO(n)$.

$\mathfrak{p}$ = symmetric matrices $\mathfrak{a}$ = diag traceless. $\mathfrak{m} = \{0\}$ (commutes w/ diag $\Rightarrow$ diag)

$\Phi = e_i - e_j \quad e_i(d_1, \dots, d_n) = d_i \quad \mathfrak{g}_{e_i - e_j} = \begin{array}{c} \\ \left(\begin{array}{ccc} & \vdots & \\ \cdot & * & \cdots \\ & \vdots & \end{array} \right) i \\ j \end{array}$ $\quad *$ any real.

$\mathfrak{q}_S$ = Block-upper triang $(\mathbb{R})$

What these give for $SL_n\mathbb{C}$, $SU(n)$

$\mathfrak{p}$ = matrices w/ $\bar{X}^t = X$ $\quad \mathfrak{a}$ = diag real traceless $\quad \mathfrak{m}$ = diag pure imag traceless

$\mathfrak{q}_S$ = Block upper triang $(\mathbb{C})$.

<u>Suggestion</u>. Work out what these give for $SU(1,1)/S^1$ or $SO(1,2)/SO(2)$.

<u>Thm</u> parab $\iff$ is the stabilizer of a point in $\partial_{vis}(G/K)$.

<u>Relation to SVD</u> M invertible.

$M = UDV$ where U, V orthog, D can be taken diag w/ weakly decr. real entries. Then typically unique. D unique.

Cor: $M \in U \exp(\mathfrak{a}) U^{-1} \cdot x_0$ where $x_0 = K$ as a pt in G/K

$K = SO(n)$ $G = SL_n\mathbb{R}$.

Similar: $G = KAK$ $A = \exp(\mathfrak{a})$. Then if $g = k_1 \exp(t) k_2$, $k_1 \exp(\mathfrak{a}) \cdot x_0$ is a flat through $x_0, g \cdot x_0$.

Typically unique if we also demand $g = k_1 \exp(t) k_2$ where $\alpha_i(t) \geq 0$ for all $\alpha_i \in \Delta$ ($\Rightarrow \forall\, \alpha_i \in \Phi^+$)

<u>Chambers</u>. $\mathfrak{a}^+ = \{ x \in \mathfrak{a} \mid \alpha(x) > 0 \ \forall\, \alpha \in \Delta \}$ Open Weyl ch.

$\mathfrak{a}^+_S = \{ x \in \mathfrak{a} \mid \alpha(x) \geq 0 \ \forall\, \alpha \in \Delta, \ \alpha(x) = 0 \text{ iff } \alpha \in S \}$

$\overline{\mathfrak{a}^+} = \{ \ \ldots \geq 0 \}$ closed Weyl chamber $= \bigcup_{S \subset \Delta} \mathfrak{a}^+_S$

<u>Cartan proj</u>. $\mu: G \to \overline{\mathfrak{a}^+}$ $\mu(g) = t$ where $g = k_1 \exp(t) k_2$

<u>Thm</u> cts, surj.

$$\vec{d}(gK, hK) = \mu(g^{-1}h) \in \overline{\mathfrak{a}^+}$$

<u>Thm:</u> $d_{Riem}(gK, hK) = \underbrace{\sqrt{B(v,v)}}_{= \|v\|}$ where $v = \vec{d}(gK, hK)$

The "real" stab theorem is:

specific ray, involves $S \subset \Delta$

<u>Thm</u>: Let $v \in \mathfrak{a}^+_S$, $\|v\| = 1$. Let $\gamma_v(t) = \exp(tv)$ $\pi = [\gamma_v] \in \partial_{vis}(G/K)$

Then $\mathrm{Stab}(\pi) = P_S$. specific parabolic, determined by S

Idea: positivity translates to asymptoticity preserving.

Distance between $\gamma_v(t)$ and $g \cdot \gamma_v(t)$ is bounded.
If g unipotent, $\sim e^{-ct}$.

Let's try this out in an example.

$SL_3\mathbb{R}$. $\mathfrak{a} = \{(a,b,c) \mid a+b+c=0\}$ $(\mathfrak{a}, B) \cong E^2$ if $(1,-\frac{1}{2},-\frac{1}{2})$ etc map to 3^{rd} roots unity

$\alpha = e_1 - e_2$ $\beta = e_2 - e_3$.

$\mathfrak{a}^+ = a > b > c$

$$\gamma(t) = \begin{pmatrix} e^{at} & & \\ & e^{bt} & \\ & & e^{ct} \end{pmatrix} \cdot x_0. \quad = \exp(tv)$$

Min parab $P = \left(\begin{smallmatrix} * & * & * \\ & * & * \\ & & * \end{smallmatrix} \right)$ should fix $[\gamma]$.

Let $g \in P$.

positive rootspace.

$$g \cdot \gamma(t) = \begin{pmatrix} d_1 e^{at} & * & * \\ & d_2 e^{bt} & * \\ & & d_3 e^{ct} \end{pmatrix}$$

exp small.

$$\vec{d}(\gamma(t), g\cdot\gamma(t)) = \mu(\exp(-tv)\, g \exp(tv))$$
$$= \mu(\mathrm{Ad}_{\exp(\underbrace{-tv}_{-\mathfrak{a}^+})}(g)) = \mu\left(\begin{smallmatrix} d_1 & * & * \\ & d_2 & * \\ & & d_3 \end{smallmatrix} \right) = \text{bdd!}$$

What if the distance is bounded? For $g = \exp(y)$ can show y has zero cpnts in the neg root spaces...

$\square$